Let us prove theorems in chapter 11.5. It is worth to learn the proofs of those theorems.

Theorem 1 (Theorem 11.5A)**.** *Suppose that a sequence x_n with $x_n \neq a$ has the limit a, and $\lim_{x \to a} f(x) = L$. Then, $\lim_{n \to \infty} f(x_n) = L$.*

Remind that $f(x)$ is not necessarily defined at a in the theorem.

Proof. Given $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - L| < \epsilon$ holds for $x \in (a - \delta, a + \delta) \setminus \{a\}$. Next, there exists a natural number N such that $|x_n - a| < \delta$ for $n \geq N$. Therefore, $|f(x_n) - L| < \epsilon$ holds for $n \geq N$, namely $\lim f(x_n) = L$. □

Example 2. *Let $\lim x_n = a$ and $f(x)$ be continuous at a. Then, $\lim f(x_n) = f(a)$.*

Proof. Since $f(x)$ is continuous at a, we have $\lim_{x \to a} f(x) = f(a)$. So, the above theorem implies the desired result. □

Example 3. *Let $f(x)$ be a continuous function defined on $\mathbb{R}$, and let $f(x) \leq 0$ hold for all $x \in \mathbb{Q}$. Then, $f(x) \leq 0$ holds for all $x \in \mathbb{R}$.*

Proof. Given a real number x and a natural number n, we choose a rational number $r_n \in (a - \frac{1}{n}, a + \frac{1}{n})$. Then, we have $\lim r_n = a$. Since $f(x)$ is continuous, we have $\lim_{n \to \infty} f(r_n) = f(a)$ by the example above.

On the other hand, $f(r_n) \leq 0$ for all n, because $r_n \in \mathbb{Q}$. Thus, the limit location theorem for sequences gives $f(x) = \lim f(r_n) \leq 0$. □

Theorem 4 (Theorem 11.5B)**.** *Let $f(x)$ be defined for $x \in (a - \delta_0, a + \delta_0) \setminus \{a\}$. Suppose that for any sequence $\{x_n\}_{n \geq 0}$ with $x_n (a - \delta_0, a + \delta_0) \setminus \{a\}$ and $\lim x_n = a$, we have $\lim f(x_n) = L$. Then, $\lim_{x \to a} f(x) = L$ holds.*

Proof. Assume that $f(x)$ does not converge to L as $x \to a$, namely diverges or converges to another number. Then, by definition of the limit, there exists $\epsilon > 0$ such that given any $\delta \in (0, \delta_0)$, $|f(x) - L| \geq \epsilon$ holds for some number $x \in (a - \delta, a + \delta) \setminus \{a\}$. Hence, for all natural number n with $\frac{1}{n} < \delta_0$, there exists a number $x_n \in (a - \frac{1}{n}, a + \frac{1}{n}) \setminus \{a\}$ such that $|f(x_n) - L| \geq \epsilon$. However, we have $\lim f(x_n) = L$ because $\lim x_n = a$. They are contradict. □

Example 5. *Let $f(x)$ be defined for $x \approx a$. Suppose that for any sequence $\{x_n\}_{n \geq 0}$ with $\lim x_n = a$, we have $\lim f(x_n) = f(a)$. Then, $f(x)$ is continuous at a.*

Proof. The Theorem 11.5B and definition of limit yield the desired result. □

Example 6. *Let $f(x)$ be defined for $x \in (a, a+\delta_0)$. Suppose that for any sequence $\{x_n\}_{n\geq 0}$ with $x_n \in (a, a+\delta_0)$ and $\lim x_n = a$, we have $\lim f(x_n) = L$. Then, $\lim_{x\to a^+} f(x) = L$ holds.*

Proof. Assume that $f(x)$ does not converge to L as $x \to a^+$. Then, there exist some $\epsilon > 0$ such that for each natural number n with $\frac{1}{n} < \delta_0$ we can choose a number $x_n \in (a, a+\frac{1}{n})$ satisfying $|f(x_n) - L| \geq \epsilon$. However, we have $\lim f(x_n) = L$ because $\lim x_n = a$. They are contradict. □

Example 7. *We define $f(x) = \int_x^1 \sin(1/t)dt$ for $x \in (0,1)$. Then, $f(x)$ is right-continuous at 0.*

Proof. For $0 < x \leq y < 1$, we have

$$(1) \quad |f(x) - f(y)| = |\int_x^y \sin(1/t)dt| \leq \int_x^y |\sin(1/t)|dt \leq \int_x^y dt = |y - x|.$$

Suppose that a sequence $\{y_n\}$ satisfies $y_n \in (0,1)$ and $\lim y_n = 0$. Then, given $\epsilon > 0$, there exists a large number N such that $|y_n| < \epsilon/2$ holds for $n \geq N$. Therefore, $|y_n - y_m| \leq |y_n| + |y_m| < \epsilon$ holds for all $n, m \geq N$. Hence, combining with (1) yields

$$|f(y_n) - f(y_m)| \leq |y_n - y_m| < \epsilon,$$

for $n, m \geq N$, namely $\{f(y_n)\}$ is a Cauchy sequence. We denote by L the limit of $\{f(y_n)\}$.

Given $\epsilon \in (0,1)$, we have $|f(y_n) - L| < \epsilon/2$ for $n \gg 1$. Since $|y_n| < \epsilon/2$ for $n \gg 1$, there exist some term y_N of the sequence $\{y_n\}$ such that $y_N \in (0, \epsilon/2)$ and $|f(y_N) - L| < \epsilon/2$. Then, for any $x \in (0, \epsilon/2)$ the following holds

$$|f(x) - L| \leq |f(x) - f(y_N)| + |f(y_N) - L| < |x - y_L| + \frac{\epsilon}{2} \leq \epsilon.$$

Therefore, $\lim_{x\to 0^+} f(x) = L$. □

Exercise 8. *Prove Example 7 by using Example 6 as follows:*

(1) *For any sequence $\{x_n\}$ with $\lim x_n = 0$, $\{f(x_n)\}$ is a Cauchy sequence and thus has the limit, as the proof above.*
(2) *Given two sequences $\{x_n\}$ and $\{y_n\}$ with $\lim x_n = \lim y_n = 0$, the limits of $\{f(x_n)\}$ and $\{f(y_n)\}$ are the same.*
(3) *Applying the result of Example 6 proves Example 7.*